

Density-shear instability in electron MHD

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We discuss a novel instability in inertia-less electron magneto-hydrodynamics (EMHD), which arises from a combination of electron velocity shear and electron density gradients. The unstable modes have a lengthscale longer than the transverse density scale, and a growth-rate of the order of the inverse Hall timescale. We suggest that this density-shear instability may be of importance in magnetic reconnection regions on scales smaller than the ion skin depth, and in neutron star crusts. We demonstrate that the so-called Hall drift instability, previously argued to be relevant in neutron star crusts, is a resistive tearing instability rather than an instability of the Hall term itself. We argue that the density-shear instability is of greater significance in neutron stars than the tearing instability, because it generally has a faster growth-rate and is less sensitive to geometry and boundary conditions. We prove that, for uniform electron density, EMHD is “at least as stable” as regular, incompressible MHD, in the sense that any field configuration that is stable in MHD is also stable in EMHD. We present a connection between the density-shear instability in EMHD and the magneto-buoyancy instability in anelastic MHD.

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I. INTRODUCTION

Electron magneto-hydrodynamics (EMHD) is a regime of plasma physics in which positive and neutral particles are approximately immobile, so that only the dynamics of the (lighter) electrons needs to be considered^{1,2}. The flow of electrons induces magnetic fields via the Hall effect, and the magnetic field influences the electrons via the Lorentz force. EMHD was first studied in the context of laboratory plasma experiments³, where it is applicable on scales smaller than the ion skin depth, and has also been used to explain (nearly) collisionless reconnection in the solar corona and magnetotail^{4–6}.

Another important application of EMHD is in neutron stars^{7,8}. Within the outermost 1km of a neutron star, called the crust, the ions are locked into a solid lattice, and the dynamics of the magnetic field is therefore governed by the EMHD equations. The structure and topology of the magnetic field determines the pattern of radiation from the star, and thus its observational signature, thermal evolution, and spin-down timescale⁹. Furthermore, the evolution of the magnetic field in the crust can trigger radiation bursts and flares, either via internal crustal failure^{10,11} or by twisting the external magnetic field lines until they undergo fast reconnection^{12,13}.

An important question then is whether magnetic fields in EMHD have a preferred structure, and whether some field configurations are unstable. Although there have been many studies of instability and turbulence in EMHD, almost all of the known instabilities require either finite ohmic resistivity¹⁴ or finite electron inertia^{15–17}. In the crust of a neutron star, however, electron inertia is entirely negligible in comparison with the Lorentz and Coulomb forces. Furthermore, most studies of EMHD turbulence have only considered the structure of the field in spectral space^{18–22}. More recently, numerical simulations have suggested that the magnetic field in a neutron star crust evolves towards a quasi-equilibrium, “frozen-in” state on a relatively short timescale ($\lesssim 1$ Myr)^{23–28}. However, these studies are all either two-dimensional, or else neglect variations in the electron density. Whether the quasi-equilibrium states found in these studies would be dynamically stable under more realistic conditions is unknown.

Rheinhardt and Geppert²⁹, RG02 hereafter, have presented numerical results demonstrating an instability in the inertia-less EMHD equations, which they argue is caused by Hall drift in the presence of a non-uniform magnetic field, and which they called the “Hall

drift instability”. However, they find that the growth-rate of the instability is very sensitive to the choice of boundary conditions, and also seems to vanish in the limit of high electrical conductivity. These observations suggest that the instability may, in fact, be a resistive tearing mode rather than an instability of the Hall term itself. The distinction is important, because tearing instability occurs only for rather particular magnetic field configurations, which calls into question the general applicability of RG02’s results to neutron stars. Tearing instability generally occurs only if the magnetic field has a null surface, within which the ideal EMHD equations become singular.

The goal of this paper is to determine the stability properties of inertia-less EHMD equilibrium states, for both finite and infinite electrical conductivity. We also consider the effect of non-uniform electron density, which was neglected in RG02’s original model, but included in a subsequent work³⁰. This effect is almost certainly important in real neutron star crusts, in which the electron density scale-height is typically only a few percent of the crust thickness³¹. We show that electron velocity shear together with density gradients can produce an instability, which resembles an instability described originally in Ref. 3. We suggest that this density-shear instability was present in the results of Ref. 30, which explains the significant discrepancies between their results and those of RG02. We present an explicit, analytical instance of the density-shear instability, and discuss its implications for the evolution of magnetic fields in neutron star crusts.

II. THE INERTIA-LESS EMHD EQUATIONS

The basic equations describing EMHD are

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \quad (1)$$

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \quad (2)$$

$$\frac{e^2 n}{\sigma} \mathbf{v} = -\frac{1}{n} \nabla P - e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right). \quad (3)$$

These are, respectively, Faraday’s law, Ampère’s law, and the force balance for the electron fluid in Gaussian cgs units. Here, \mathbf{E} and \mathbf{B} are the electric and magnetic fields, \mathbf{J} is the electric current, \mathbf{v} is the electron fluid velocity, n is the electron number density, P is the electron pressure, σ is the electrical conductivity, c is the speed of light, and $e = |e|$ is the elementary charge. Note that we neglect the electron inertia in Equation (3), which is

negligible on scales much larger than the electron skin depth, $d = (c/e)(m^*/4\pi n)^{1/2}$, where m^* is the effective mass of the electrons. In the crust of a neutron star the skin depth is tiny, $d \sim 10^{-11}$ cm, making this an excellent approximation³¹. As noted earlier, the absence of electron inertia has important implications for EMHD stability, because most known instabilities in EMHD arise from inertial effects.

The electric current produced by the flow of electrons is

$$\mathbf{J} = -en\mathbf{v}, \quad (4)$$

and substituting this relation into Equation (3) yields a generalized Ohm's law. Equations (2) and (4) together imply that the electron density n is steady in the Eulerian description, because

$$\begin{aligned} \frac{\partial n}{\partial t} &= -\nabla \cdot (n\mathbf{v}) \\ &= \frac{c}{4\pi e} \nabla \cdot (\nabla \times \mathbf{B}) \\ &= 0. \end{aligned} \quad (5)$$

Equations (1)–(4) can be combined into a single equation for the evolution of the magnetic field,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\frac{c}{4\pi en} \mathbf{B} \times (\nabla \times \mathbf{B}) - \frac{c^2}{4\pi\sigma} \nabla \times \mathbf{B} \right] + \frac{c}{en^2} \nabla P \times \nabla n. \quad (6)$$

The three terms on the right-hand side are, respectively, the Hall effect, ohmic decay, and the Biermann battery effect. From here on, we follow Ref. 7 and RG02 in neglecting the Biermann battery term, which represents the generation of electric currents by electron baroclinicity. This term becomes negligible at sufficiently low temperatures, for which the electron gas is fully degenerate and therefore barotropic. More precisely, the Biermann term is negligible in comparison with the Hall term if

$$(T/T_F)^2 \ll \frac{|\mathbf{B}|^2}{4\pi\epsilon_F n}, \quad (7)$$

where ϵ_F is the Fermi energy and $T_F = \epsilon_F/k_B$ is the Fermi temperature. Typical orders of magnitude for the various parameters in a neutron star crust are⁸ $n \sim 10^{35} \text{ cm}^{-3}$, $\epsilon_F \sim 10^{-4} \text{ erg}$, $|\mathbf{B}| \sim 10^{13} \text{ G}$, and $\sigma \sim 10^{25} \text{ s}^{-1}$. Condition (7) is then $T \ll 10^9 \text{ K}$, which is satisfied in isolated neutron stars older than about 1 Myr.

The relative importance of the Hall and ohmic decay terms in Equation (6) is measured by the Hall parameter,

$$H = \frac{\sigma}{cen} |\mathbf{B}|. \quad (8)$$

For the parameter values just given we find $H \sim 10^2$, implying that the Hall term dominates the evolution of the magnetic field. Assuming a typical lengthscale $L \sim 1$ km, comparable to the thickness of the crust, the characteristic timescale for magnetic field evolution is then

$$t_{\text{Hall}} = \frac{4\pi enL^2}{c|\mathbf{B}|} \sim 1 \text{ Myr}. \quad (9)$$

In what follows, we work exclusively with non-dimensional quantities. We use L and t_{Hall} as units of length and time respectively, and we measure \mathbf{B} and n in units of the characteristic values given above. In these units, Equation (6) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\frac{1}{n} \mathbf{B} \times \mathbf{J} - \eta \mathbf{J} \right], \quad (10)$$

where η is a dimensionless diffusivity of order $H^{-1} \ll 1$, and where

$$\mathbf{J} = \nabla \times \mathbf{B} \quad (11)$$

is the dimensionless electric current. We will consider linear perturbations to equilibrium states, i.e., to steady solutions of Equation (10). The perturbation to the magnetic field, $\delta \mathbf{B}$, obeys the linear equation

$$\frac{\partial}{\partial t} \delta \mathbf{B} = \nabla \times \left[-\frac{1}{n} \mathbf{J} \times \delta \mathbf{B} + \frac{1}{n} \mathbf{B} \times (\nabla \times \delta \mathbf{B}) - \eta \nabla \times \delta \mathbf{B} \right]. \quad (12)$$

The first term on the right-hand side of Equation (12) represents advection of the perturbations by the electron velocity, the second term gives rise to whistler waves (a.k.a. helicons) for a uniform background field and uniform density^{32,33}, and the third term represents resistive diffusion of the perturbation.

From Equation (10) it can be shown that

$$\frac{d}{dt} \int_V dV \frac{1}{2} |\mathbf{B}|^2 = - \int_V dV \eta |\mathbf{J}|^2 - \int_{\partial V} dS \hat{\mathbf{n}} \cdot \mathbf{B} \times \left[\frac{1}{n} \mathbf{B} \times \mathbf{J} - \eta \mathbf{J} \right] \quad (13)$$

for any volume V with boundary ∂V and outward normal $\hat{\mathbf{n}}$. For a closed system the surface integral vanishes, and so the magnetic energy decays in time, at a rate that depends on the diffusivity η . This does not, however, imply that the system is stable, because perturbations

can still grow by extracting energy from the background magnetic field. In fact, it can be shown from Equation (12) that, omitting surface integrals,

$$\frac{d}{dt} \int_V dV \frac{1}{2} |\delta \mathbf{B}|^2 = \int_V dV \left[\delta B_i \left(\nabla \cdot (\mathbf{J}/n) \frac{\delta_{ij}}{2} - \frac{\partial (J_i/n)}{\partial x_j} \right) \delta B_j - \eta |\delta \mathbf{J}|^2 \right]. \quad (14)$$

So, in a closed system, the only possible source of instability is spatial gradients in \mathbf{J}/n , i.e., gradients in the electron velocity.

III. UNIFORM ELECTRON DENSITY

We begin by considering the simplest case of EMHD, in which $\eta = 0$ and n is constant. Without loss of generality we take $n = 1$ — any other constant value can be obtained simply by rescaling the magnetic field. We consider linear perturbations $\delta \mathbf{B}$ to a steady background field \mathbf{B} , which is a solution of the equation

$$0 = \nabla \times [\mathbf{B} \times (\nabla \times \mathbf{B})]. \quad (15)$$

The linear equation for the perturbations (12) simplifies in this case to

$$\frac{\partial}{\partial t} \delta \mathbf{B} = \nabla \times [-\mathbf{J} \times \delta \mathbf{B} + \mathbf{B} \times (\nabla \times \delta \mathbf{B})], \quad (16)$$

and Equation (14) becomes

$$\frac{d}{dt} \int_V dV \frac{1}{2} |\delta \mathbf{B}|^2 = - \int_V dV \delta B_i \frac{\partial J_i}{\partial x_j} \delta B_j. \quad (17)$$

So a necessary condition for instability is the presence of electron velocity shear.

A. Stability of straight field lines

Following RG02, we now adopt a Cartesian coordinate system (x, y, z) and consider a background magnetic field of the form

$$\mathbf{B} = B(z) \mathbf{e}_x \quad (18)$$

where \mathbf{e}_x is the unit vector in the x direction. A field of this form satisfies the equilibrium condition (15) for any choice of the function $B(z)$. Since the background field depends only on z , we may seek eigenmode solutions to the perturbation equation (16) of the form

$$\delta \mathbf{B} = \mathbf{b}(z) \exp(\lambda t + i k_x x + i k_y y) \quad (19)$$

where λ is the (possibly complex) growth-rate and \mathbf{b} is a complex amplitude function. Substituting (18) and (19) into Equation (16), we eventually obtain a single equation for the z component of \mathbf{b} :

$$\frac{b_z''}{b_z} = \frac{B''}{B} + k_x^2 + k_y^2 + \left(\frac{\lambda}{Bk_x} - i \frac{B'k_y}{Bk_x} \right) \frac{\lambda}{Bk_x}, \quad (20)$$

where primes denote derivatives with respect to z . For the moment, we restrict attention to perturbations that have $k_y = 0$. This equation then becomes simply

$$\frac{b_z''}{b_z} = \frac{B''}{B} + k_x^2 + \frac{\lambda^2}{B^2 k_x^2}. \quad (21)$$

By analogy with the one-dimensional Schrödinger equation for a potential well, we see that in order for this equation to have bounded solutions the right-hand side must be negative for some range of z . Therefore a necessary condition for instability (i.e., $\text{Re}\{\lambda\} > 0$) is that the term B''/B must be negative for some range of z . RG02 argue that there will be unstable modes provided that B''/B is chosen to be sufficiently negative (see also Ref. 8, §4.3.5). However, this reasoning is flawed, as can be seen by multiplying Equation (21) by $|b_z|^2$ and integrating in z . We then obtain

$$\left[b_z^* \left(b_z' - \frac{B'}{B} b_z \right) \right] = \int dz \left| b_z' - \frac{B'}{B} b_z \right|^2 + \int dz \left(k_x^2 + \frac{\lambda^2}{B^2 k_x^2} \right) |b_z|^2, \quad (22)$$

where b_z^* denotes the complex conjugate of b_z . If the boundary conditions are chosen so that the left-hand side of Equation (22) vanishes, then we deduce that

$$-\lambda^2 = \frac{k_x^2 \int dz \left| b_z' - \frac{B'}{B} b_z \right|^2 + k_x^4 \int dz |b_z|^2}{\int dz \frac{|b_z|^2}{B^2}}, \quad (23)$$

implying that the growth-rate λ is purely imaginary, and so all modes are neutrally stable. This conclusion also applies for practically any other sensible choice of boundary conditions. For example, suppose that our domain is $-1 < z < 1$, and that the region outside the domain is a vacuum. The background field must then have $B'(z) = 0$ at $z = \pm 1$, and the boundary conditions for the perturbations are $b_z' = \mp |k_x| b_z$ at $z = \pm 1$. The left-hand side of Equation (22) is then negative, and the conclusion that λ is imaginary holds even more strongly. Allowing for $k_y \neq 0$ does not alter this conclusion; in fact, it can be shown from

Equation (20) that the generalization of Equation (23) when $k_y \neq 0$ is

$$-\left[\lambda - \frac{1}{2}ik_y \frac{\int dz \frac{B'}{B^2} |b_z|^2}{\int dz \frac{|b_z|^2}{B^2}}\right]^2 = \frac{k_x^2 \int dz \left|b'_z - \frac{B'}{B} b_z\right|^2 + k_x^2 (k_x^2 + k_y^2) \int dz |b_z|^2}{\int dz \frac{|b_z|^2}{B^2}} + \left[\frac{1}{2}k_y \frac{\int dz \frac{B'}{B^2} |b_z|^2}{\int dz \frac{|b_z|^2}{B^2}}\right]^2. \quad (24)$$

So k_y produces a frequency splitting between modes that propagate “upstream” and “downstream” with respect to the electron velocity, but all modes remain purely oscillatory, and therefore the system is stable.

It is straightforward to generalize these results still further by considering a magnetic field of the form $\mathbf{B} = B_x(z) \mathbf{e}_x + B_y(z) \mathbf{e}_y$, which automatically satisfies the equilibrium condition (15). Equation (20) then becomes

$$\frac{b''_z}{b_z} = \frac{\mathbf{B}'' \cdot \mathbf{k}}{\mathbf{B} \cdot \mathbf{k}} + |\mathbf{k}|^2 + \left(\frac{\lambda}{\mathbf{B} \cdot \mathbf{k}} - i \frac{[\mathbf{B}' \times \mathbf{k}]_z}{\mathbf{B} \cdot \mathbf{k}} \right) \frac{\lambda}{\mathbf{B} \cdot \mathbf{k}}, \quad (25)$$

from which stability can be demonstrated as before. In this way, it can be shown that any field with straight field lines (i.e., with $\mathbf{B} \cdot \nabla \mathbf{B} = 0$) is stable in EMHD if the electron density is uniform. This result can also be obtained by a more mathematically elegant argument, as shown later in §III C.

We note that this result contradicts Ref. 34, which claimed to have demonstrated instability for a magnetic field with straight field lines. Although their derivation assumes finite electron inertia, their instability remains even in the limit of zero inertia. We suggest that there are three reasons why they obtained this incorrect result. First, they assumed a background magnetic field that varies in one direction, but they only considered perturbations that are invariant in that direction. This is equivalent to seeking solutions of Equation (25) for which the left-hand side vanishes, which is obviously impossible for any non-trivial magnetic field. Second, they prescribed a uniform background magnetic field and a non-uniform background current, which is incompatible with Ampère’s law (2). Third, they considered a localized region in which the current changes sign, and therefore neglected terms involving the current while retaining terms involving its spatial gradient. There is no rigorous basis for this approximation. We believe that these inconsistencies explain the contradiction between their results and ours. We emphasize, however, that there *are* instabilities in the EMHD equations when the fluid has finite inertia. This can be proved by simply observing that, on scales much smaller than the electron skin depth, the EMHD equations become identical to

the equations for an incompressible, non-magnetic fluid^{16,17}, for which there are numerous well-known instabilities.

B. Resistive tearing instability

The argument presented in the last section assumes, implicitly, that all of the integrals in Equations (22)–(24) are well defined. However, if the background field in Equation (18) is chosen such that $B(z)$ vanishes somewhere within the domain then these integrals may be singular. We see from Equations (20) and (21) that the eigenmodes are also singular in such cases. This suggests that reintroducing finite resistivity, $\eta > 0$, will significantly alter the structure of the eigenmodes, as well as their stability properties. In particular, by analogy with “regular” MHD, we anticipate that such fields can be subject to resistive tearing instabilities^{35,36}. The existence of tearing instabilities in EMHD has been convincingly demonstrated previously^{14,15,37}, so here we simply summarize the essential points of the analysis, and compare the predictions of the theory with the results of RG02. For simplicity, we also restrict attention to modes of the form given by Equation (19) with $k_y = 0$; the fastest growing mode found by RG02 was in this category.

Suppose that $B(z)$ vanishes for some value of z , say $z = z_0$. If resistive diffusion is sufficiently weak then we expect Equation (21) to hold to a good approximation away from the singularity at z_0 . We refer to the solution of this equation as the “outer” solution. In a neighborhood of the singularity we approximate $B(z) = B'(z_0)(z - z_0)$, and we assume that η is constant. Substituting the ansatz (19) into Equation (12), we thus obtain a pair of coupled equations for b_y and b_z ,

$$(\lambda - \eta \nabla^2) b_y = B'(z - z_0) \nabla^2 b_z \quad (26)$$

$$(\lambda - \eta \nabla^2) b_z = B'(z - z_0) k_x^2 b_y, \quad (27)$$

whose solution we will call the “inner” solution. For now, we assume that B' and k_x are of order unity, and that η is of order $H^{-1} \ll 1$. We then find that resistive diffusion only becomes important on scales smaller than $\sqrt{\eta/|B'k_x|} \sim H^{-1/2}$, and we therefore approximate $\nabla^2 \simeq \partial^2/\partial z^2$ in these equations. We seek solutions for which b_y is antisymmetric about z_0 and b_z is symmetric. Following Ref. 35 we anticipate that, provided $|\lambda| \ll |B'k_x|$, both b_y and b_z will be approximately constant in a neighborhood of z_0 . This allows us to

approximate³⁸ Equations (26) and (27) as

$$-\eta \frac{\partial^2}{\partial z^2} b_y = B'(z - z_0) \frac{\partial^2}{\partial z^2} b_z \quad (28)$$

$$\lambda b_{z0} - \eta \frac{\partial^2}{\partial z^2} b_z = B'(z - z_0) k_x^2 b_y, \quad (29)$$

where b_{z0} is the value of b_z at $z = z_0$. Finally, we introduce a stretched coordinate $\tilde{z} = \left| \frac{2B'k_x}{\eta} \right|^{1/2} (z - z_0)$, which leads to the following equation for the quantity $b(\tilde{z}) = \frac{B'}{\lambda b_{z0}} \left| \frac{2k_x^3 \eta}{B'} \right|^{1/2} b_y$:

$$b'' - \frac{1}{4} \tilde{z}^2 b + \frac{1}{2} \tilde{z} = 0. \quad (30)$$

This is a special case of the equation obtained in Ref. 35, and so we can use their results from here on. The unique regular solution of Equation (30) can be expressed as a sum of Hermite functions,

$$b(\tilde{z}) = \sum_{\text{odd } n} \frac{2}{n + \frac{1}{2}} \left[\frac{\Gamma(\frac{1}{2}n + 1)}{\Gamma(\frac{1}{2}n + \frac{1}{2})} \right]^{1/2} \frac{(-1)^n e^{\tilde{z}^2/4}}{\pi^{1/4} (n!)^{1/2}} \frac{d^n}{d\tilde{z}^n} e^{-\tilde{z}^2/2}. \quad (31)$$

Although b_z is roughly constant in the inner solution, its z derivative changes rapidly across $z = z_0$, by an amount

$$\begin{aligned} \Delta' &\equiv \frac{1}{b_{z0}} \int dz \frac{\partial^2}{\partial z^2} b_z \\ &= \frac{\lambda}{\sqrt{|2B'k_x\eta|}} \int_{-\infty}^{+\infty} d\tilde{z} (1 - \frac{1}{2} \tilde{z} b(\tilde{z})) \\ &= 2\pi \frac{\Gamma(3/4)}{\Gamma(1/4)} \frac{\lambda}{\sqrt{|B'k_x\eta|}}. \end{aligned} \quad (32)$$

Instability therefore requires that $\Delta' > 0$. The approximations made in Equations (28) and (29) are self-consistent provided that $1/|\Delta'|$ is much larger than the width of the inner region, i.e.,

$$\begin{aligned} \left| \frac{\eta}{B'k_x} \right|^{1/2} &\ll 1/|\Delta'| \\ \Leftrightarrow |\lambda| &\ll |B'k_x| \end{aligned}$$

as expected.

RG02 take a background field of the form $B(z) = B_0(1 - z^2)$, with B_0 a constant, over the domain $-1 < z < 1$, and use “vacuum” boundary conditions for the perturbations, i.e.,

$b_y = 0$ and $\frac{\partial}{\partial z}b_z = \mp|k_x|b_z$ at $z = \pm 1$. Because the singularities in their case occur exactly at the boundaries of the domain, the change in the derivative of b_z across the inner regions is $\frac{1}{2}\Delta'$. The effective boundary conditions for the outer solution in their case are therefore

$$\frac{\partial}{\partial z}b_z = \mp(|k_x| + \frac{1}{2}\Delta')b_z \quad \text{at } z = \pm 1. \quad (33)$$

We are now in a position to calculate the dispersion relation for the EMHD tearing modes. Rather than taking RG02's field $B(z) = B_0(1 - z^2)$, we use a very similar field $B(z) = B_0 \cos(\frac{\pi}{2}z)$ that allows us to calculate the outer solution analytically. The general solution of Equation (21) can then be written in terms of associated Legendre functions,

$$b_z = \cos^{1/2}(\frac{\pi}{2}z)P_l^m(\sin(\frac{\pi}{2}z)), \quad (34)$$

where
$$m^2 = \frac{1}{4} + \left(\frac{2\lambda}{\pi B_0 k_x}\right)^2 \quad (35)$$

and
$$\left(l + \frac{1}{2}\right)^2 = 1 - \left(\frac{2k_x}{\pi}\right)^2. \quad (36)$$

The only outer solutions that are bounded for all z are those with $m \leq \frac{1}{2}$, implying that the growth-rate λ would be imaginary in the absence of singularities. We now seek tearing solutions, which are marginally stable (i.e., $\lambda = 0$) outer solutions that can be matched to the boundary conditions (33) with $\Delta' > 0$. The marginally stable outer solutions form odd and even families, $b_z = \sin(\frac{\pi^2}{4} - k_x^2)^{1/2}z$ and $b_z = \cos(\frac{\pi^2}{4} - k_x^2)^{1/2}z$ respectively. Applying the boundary conditions (33) leads to distinct dispersion relations for the two families:

$$\text{odd: } \frac{\Gamma(3/4)}{\Gamma(1/4)} \left(\frac{2\pi}{\eta B_0 |k_x|}\right)^{\frac{1}{2}} \lambda = - \left(\frac{\pi^2}{4} - k_x^2\right)^{\frac{1}{2}} \cot \left(\frac{\pi^2}{4} - k_x^2\right)^{\frac{1}{2}} - |k_x| \quad (37)$$

$$\text{even: } \frac{\Gamma(3/4)}{\Gamma(1/4)} \left(\frac{2\pi}{\eta B_0 |k_x|}\right)^{\frac{1}{2}} \lambda = \left(\frac{\pi^2}{4} - k_x^2\right)^{\frac{1}{2}} \tan \left(\frac{\pi^2}{4} - k_x^2\right)^{\frac{1}{2}} - |k_x|. \quad (38)$$

The odd modes are stable for all k_x , but the even modes are unstable for $k_x \lesssim 1$. Away from the boundaries, the structure of the unstable modes, given by the outer solution, is

$$\delta \mathbf{B} \propto \begin{pmatrix} (\frac{\pi^2}{4} - k_x^2)^{1/2} \\ 0 \\ ik_x \cot \left[(\frac{\pi^2}{4} - k_x^2)^{1/2} z \right] \end{pmatrix} \sin \left[(\frac{\pi^2}{4} - k_x^2)^{1/2} z \right] \exp(ik_x x + \lambda t) \quad (39)$$

so the instability is purely two-dimensional, except near the boundaries, and undular in x . The growth-rate given by Equation (38) diverges as $|k_x| \rightarrow 0$, with $\lambda \sim (\eta B_0)^{1/2} |k_x|^{-3/2}$, but

the self-consistency condition $\lambda \ll B_0|k_x|$ is evidently violated in this limit. We expect the actual fastest growing mode to have $\lambda \sim B_0|k_x| \sim (\eta B_0)^{1/2}|k_x|^{-3/2}$ in the limit $(\eta/B_0) \sim H^{-1} \rightarrow 0$, and so $\lambda \sim \eta^{1/5}B_0^{4/5}$ and $|k_x| \sim (\eta/B_0)^{1/5}$.

Although we are not able to analytically predict the exact wavenumber and growth-rate for the background field studied by RG02, we can make the following predictions regarding the most unstable tearing mode in their case:

1. in the bulk of the domain, the instability is undular and two-dimensional, with a wavenumber $\sim (\eta/B_0)^{1/5}$ in the field-wise direction;
2. the eigenmode adjusts rapidly to meet the boundary conditions, within a layer of width $\sim (\eta/B_0)^{2/5}$;
3. the growth-rate λ vanishes as $\eta \rightarrow 0$, roughly as $\eta^{1/5}B_0^{4/5}$;
4. the growth-rate is very sensitive to boundary conditions.

The instability found numerically by RG02 agrees with each of these predictions. In particular, they find that $\delta B_y \simeq 0$, except in thin boundary layers, and that the growth-rate vanishes in the limit $\eta \rightarrow 0$ with $\lambda \sim \eta(B_0/\eta)^q$ and $q \in (0.7, 0.9)$. These are strong indications that the instability is driven by diffusion across the singularities where $|\mathbf{B}| = 0$, rather than by the Hall effect itself.

In the light of the above, we might question whether the instability found by RG02 has much relevance to neutron stars. The tearing instability requires rather specific conditions, and it is not clear that these would arise naturally. In fact, the particular choice of background magnetic field $\mathbf{B} = B_0(1 - z^2)\mathbf{e}_x$ used by RG02 is rather artificial, and does not have an obvious analogue in spherical geometry. In their model, z is intended to be the vertical coordinate, and therefore becomes the radial coordinate in spherical geometry. Because there are no spherical EMHD equilibrium states with a purely toroidal field³⁹, we must interpret x as the latitudinal coordinate. But in a sphere with vacuum outer boundary conditions, the latitudinal component of the field would not be expected to vanish at the outer boundary, and so there would be no tearing instability there. Nor is the latitudinal component expected to vanish at the bottom of the crust, at the boundary with the superconducting outer core.

C. Connection with incompressible MHD

If the instability found by RG02 is in fact a resistive tearing instability, then there remains the question of whether there exist any instabilities in ideal (i.e., non-resistive) inertia-less EMHD. In analyzing this question, it proves useful to use a new linear perturbation variable in place of $\delta\mathbf{B}$. We therefore introduce the Lagrangian perturbation $\boldsymbol{\xi}$, which we define as the difference between the Eulerian position of a Lagrangian particle in the perturbed and unperturbed systems. The (Eulerian) perturbations to the magnetic field and electron density can then be expressed as

$$\delta\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}) \quad (40)$$

$$\delta n = -\boldsymbol{\nabla} \cdot (n\boldsymbol{\xi}). \quad (41)$$

The Lagrangian perturbation is kinematically related to the electron velocity as

$$\frac{\partial}{\partial t}(n\boldsymbol{\xi}) = \delta(n\mathbf{v}) + \boldsymbol{\nabla} \times (n\mathbf{v} \times \boldsymbol{\xi}). \quad (42)$$

We close the equations using the (dimensionless) relations

$$\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{B} \quad \Rightarrow \quad \delta\mathbf{J} = \boldsymbol{\nabla} \times \delta\mathbf{B} \quad (43)$$

$$\mathbf{J} = -n\mathbf{v} \quad \Rightarrow \quad \delta\mathbf{J} = -\delta(n\mathbf{v}) \quad (44)$$

which imply that

$$n \frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\mathbf{B} \times \boldsymbol{\xi}) - \mathbf{J} \times \boldsymbol{\xi}] \quad (45)$$

$$\boldsymbol{\nabla} \cdot (n\boldsymbol{\xi}) = 0. \quad (46)$$

Taking the background density to be $n = 1$, the linear equations for $\boldsymbol{\xi}$ become

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\mathbf{B} \times \boldsymbol{\xi}) - \mathbf{J} \times \boldsymbol{\xi}] \quad (47)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0. \quad (48)$$

We note that the linear equation for $\delta\mathbf{B}$ (16) can be recovered from Equation (47) using the relation (40). Finally, we note that the perturbation to the Lorentz force is

$$\begin{aligned} \delta(\mathbf{J} \times \mathbf{B}) &= \delta\mathbf{J} \times \mathbf{B} + \mathbf{J} \times \delta\mathbf{B} \\ &= \mathcal{F}(\boldsymbol{\xi}), \end{aligned} \quad (49)$$

where \mathcal{F} is the linear operator

$$\mathcal{F}(\boldsymbol{\xi}) = \mathbf{B} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} \times \boldsymbol{\xi} - \mathbf{J} \times \boldsymbol{\nabla} \times \mathbf{B} \times \boldsymbol{\xi}. \quad (50)$$

Significantly, the operator \mathcal{F} is self-adjoint over the space of (complex) vector fields satisfying the constraint (48) and with respect to the inner product $\langle \cdot, \cdot \rangle$ defined as

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \equiv \int dV \boldsymbol{\alpha}^* \cdot \boldsymbol{\beta}, \quad (51)$$

where $\boldsymbol{\alpha}^*$ denotes the complex conjugate of $\boldsymbol{\alpha}$. To prove the self-adjointness of \mathcal{F} , we make use of the identity

$$\begin{aligned} & \boldsymbol{\nabla}(\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \times \boldsymbol{\gamma}) + (\boldsymbol{\alpha} \times \boldsymbol{\beta}) \boldsymbol{\nabla} \cdot \boldsymbol{\gamma} + (\boldsymbol{\beta} \times \boldsymbol{\gamma}) \boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + (\boldsymbol{\gamma} \times \boldsymbol{\alpha}) \boldsymbol{\nabla} \cdot \boldsymbol{\beta} = \\ & \boldsymbol{\alpha} \times (\boldsymbol{\nabla} \times (\boldsymbol{\beta} \times \boldsymbol{\gamma})) + \boldsymbol{\beta} \times (\boldsymbol{\nabla} \times (\boldsymbol{\gamma} \times \boldsymbol{\alpha})) + \boldsymbol{\gamma} \times (\boldsymbol{\nabla} \times (\boldsymbol{\alpha} \times \boldsymbol{\beta})), \end{aligned} \quad (52)$$

which holds for any triple $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$. If the background field \mathbf{B} satisfies the equilibrium condition (15), then it follows that

$$\boldsymbol{\nabla}(\boldsymbol{\alpha} \cdot \mathbf{B} \times \mathbf{J}) + (\mathbf{B} \times \mathbf{J}) \boldsymbol{\nabla} \cdot \boldsymbol{\alpha} = \mathbf{B} \times (\boldsymbol{\nabla} \times (\mathbf{J} \times \boldsymbol{\alpha})) + \mathbf{J} \times (\boldsymbol{\nabla} \times (\boldsymbol{\alpha} \times \mathbf{B})). \quad (53)$$

From this result, and using integration by parts repeatedly, we find that

$$\begin{aligned} \langle \boldsymbol{\alpha}, \mathcal{F}(\boldsymbol{\beta}) \rangle - \langle \mathcal{F}(\boldsymbol{\alpha}), \boldsymbol{\beta} \rangle &= \int dV \boldsymbol{\nabla} \cdot [(\mathbf{B} \times \boldsymbol{\alpha}^*) \times (\boldsymbol{\nabla} \times \mathbf{B} \times \boldsymbol{\beta}) \\ &\quad - (\mathbf{B} \times \boldsymbol{\beta}) \times (\boldsymbol{\nabla} \times \mathbf{B} \times \boldsymbol{\alpha}^*) - \mathbf{J} \cdot (\boldsymbol{\alpha}^* \times \boldsymbol{\beta}) \mathbf{B}] \end{aligned} \quad (54)$$

for any two divergence-free vector fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Therefore if $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are two solutions of Equations (47)–(48), and if boundary conditions are chosen appropriately, then

$$\langle \boldsymbol{\xi}_1, \mathcal{F}(\boldsymbol{\xi}_2) \rangle = \langle \mathcal{F}(\boldsymbol{\xi}_1), \boldsymbol{\xi}_2 \rangle, \quad (55)$$

demonstrating that \mathcal{F} is indeed self-adjoint. It can also be shown that

$$\begin{aligned} \frac{d}{dt} \langle \boldsymbol{\xi}_1, \mathcal{F}(\boldsymbol{\xi}_2) \rangle &= \int dV \boldsymbol{\nabla} \cdot \left[(\mathbf{B} \times \boldsymbol{\xi}_1^*) \times \boldsymbol{\nabla} \times \mathbf{B} \times \frac{\partial \boldsymbol{\xi}_2}{\partial t} + (\boldsymbol{\nabla} \times \mathbf{B} \times \boldsymbol{\xi}_1^*) \times \mathbf{B} \times \frac{\partial \boldsymbol{\xi}_2}{\partial t} \right. \\ &\quad \left. - (\mathbf{J} \times \boldsymbol{\xi}_1^*) \times \mathbf{B} \times \frac{\partial \boldsymbol{\xi}_2}{\partial t} - (\mathbf{J} \times \mathbf{B} \cdot \boldsymbol{\xi}_2) \frac{\partial \boldsymbol{\xi}_1^*}{\partial t} \right], \end{aligned} \quad (56)$$

so the quantity $\langle \boldsymbol{\xi}_1, \mathcal{F}(\boldsymbol{\xi}_2) \rangle$ is conserved for suitable boundary conditions. This implies, in particular, that if $\boldsymbol{\xi}$ is an eigensolution of Equation (47) with growth-rate λ then

$$\frac{d}{dt} \langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle = (\lambda + \lambda^*) \langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle = 0. \quad (57)$$

Therefore any unstable mode ($\lambda + \lambda^* > 0$) must be a solution of the equation $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle = 0$. Since \mathcal{F} is a self-adjoint operator, a necessary condition for instability is therefore that \mathcal{F} has both positive and negative eigenvalues. Furthermore, it can be shown from Equation (50) that

$$\mathcal{F}(\boldsymbol{\xi}) = (\mathbf{B} \cdot \nabla)^2 \boldsymbol{\xi} - \nabla(\mathbf{B} \cdot \delta \mathbf{B}) - \boldsymbol{\xi} \cdot \nabla(\mathbf{B} \cdot \nabla \mathbf{B}) - (\nabla \cdot \boldsymbol{\xi}) \mathbf{B} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla[(\nabla \cdot \boldsymbol{\xi}) \mathbf{B}]. \quad (58)$$

From this, and using the fact that any EMHD equilibrium must have

$$\mathbf{B} \cdot \nabla \mathbf{B} = \nabla \psi \quad (59)$$

for some function ψ , it follows that

$$\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle = - \int dV \left[|\mathbf{B} \cdot \nabla \boldsymbol{\xi}|^2 + \xi_i \frac{\partial^2 \psi}{\partial x_i \partial x_j} \xi_j^* \right]. \quad (60)$$

We deduce immediately the result mentioned in §III A, that a field with $\mathbf{B} \cdot \nabla \mathbf{B} = 0$ is always stable, since then the right-hand side of Equation (60) is always negative.

The quantity in Equation (60) arises in another context — that of “regular” MHD for an incompressible fluid⁴⁰. In that context, the necessary *and sufficient* condition for instability of a static equilibrium is that $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle$ is positive for some perturbation $\boldsymbol{\xi}$. This is because $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle$ represents the change in magnetic energy produced by the perturbation (to second order in $\boldsymbol{\xi}$), so a positive eigenvalue of the operator \mathcal{F} represents a perturbation that converts energy stored in the magnetic field into kinetic energy. In EMHD, on the other hand, the total magnetic energy is always conserved, and the inertia-less electron fluid has no kinetic energy. The only perturbations that can grow, therefore, are those that do not change the total magnetic energy of the system, i.e., those that have $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle = 0$.

In summary, the necessary and sufficient condition for instability of a static magnetic field in incompressible MHD is a necessary condition for instability in EMHD. This means that EMHD is “at least as stable” as incompressible MHD, because all field configurations that are stable in MHD are also stable in EMHD. In fact, we can say that EMHD is *more* stable than incompressible MHD, because there are fields that are unstable in MHD but stable in EMHD. For example, a purely toroidal field of the form $\mathbf{B} = \mathbf{e}_z \times \mathbf{x}$ is subject to “kink” instability in MHD⁴¹, but is stable in EMHD, as can be deduced immediately from Equation (17).

IV. NON-UNIFORM ELECTRON DENSITY

In a neutron star crust the electron density varies by several orders of magnitude³¹, with a typical density scale-height of 10^3 – 10^4 cm. Therefore, the consequences of density gradients for EMHD stability must be considered in any realistic model of neutron star fields. In this section we extend our previous results to take account of such gradients. To simplify the analysis we will neglect resistivity from here on, and so our basic equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\frac{1}{n} \mathbf{B} \times (\nabla \times \mathbf{B}) \right], \quad (61)$$

where the electron density $n(\mathbf{x})$ is prescribed, and we consider linear perturbations $\delta \mathbf{B}(\mathbf{x}, t)$ to a steady equilibrium solution $\mathbf{B}(\mathbf{x})$.

A. Density-shear instability

We begin with the simplest example of an EMHD equilibrium field with non-uniform density, which is $\mathbf{B} = B(z) \mathbf{e}_x$ and $n = n(z)$. To avoid any issues arising from singularities and boundary conditions, we will assume from here on that the domain is infinite, and that $B(z)$ and $n(z)$ are strictly positive for all z and remain bounded as $|z| \rightarrow \infty$. As in §III A, we seek magnetic field perturbations of the form (19). In place of Equation (20) we now find

$$\frac{b_z''}{b_z} = \frac{(B'/n)'}{B/n} + k_x^2 + k_y^2 + \left(\frac{\lambda n}{Bk_x} - i \frac{B'k_y}{Bk_x} \right) \left(\frac{\lambda n}{Bk_x} - i \frac{n'k_y}{nk_x} \right). \quad (62)$$

As with Equation (20), a necessary condition for instability is that the first term on the right-hand side is negative for some range of z . We note that the (dimensionless) electron velocity in the background state is $\mathbf{v} = -(\nabla \times \mathbf{B})/n = -(B'/n)\mathbf{e}_y$, so instability requires the presence of electron shear⁸. However, as in §III A, this condition is only necessary, and not sufficient. A more stringent necessary condition can be obtained by the same process that led to Equation (23), which this time leads to the result

$$\lambda^2 = \frac{k_x^2 \int dz \frac{B'n'}{Bn} |b_z|^2 - k_x^2 \int dz \left| b_z' - \frac{B'}{B} b_z \right|^2 - k_x^4 \int dz |b_z|^2}{\int dz \frac{n^2}{B^2} |b_z|^2} \quad (63)$$

for perturbations with $k_y = 0$. So a necessary condition for instability is that $\frac{B'n'}{Bn} > 0$ for some range of z . A similar result was obtained in Ref. 3 for instabilities of a magneto-sonic

wavefront. However, they were only able to demonstrate instability in the limit where the width of the front becomes much narrower than the wavelength of the perturbations. In fact, instabilities can be found under much more general conditions. For example, suppose that $B(z) = n(z) = \text{sech}^\gamma(z)$ for some positive constant γ . Then the general solution of Equation (62) can be expressed in terms of Jacobi polynomials:

$$b_z = \text{sech}^\alpha(z) \exp(-i\beta z) P_m^{(\alpha+i\beta, \alpha-i\beta)}(\tanh(z)) \quad (64)$$

where the parameters α , β , and m are related to k_x , k_y , and λ by the equations

$$\alpha\beta = \gamma\lambda k_y/k_x^2 \quad (65)$$

$$\alpha^2 - \beta^2 = k_x^2 + k_y^2 + (\lambda^2 - \gamma^2 k_y^2)/k_x^2 \quad (66)$$

$$(\alpha + m)(\alpha + m + 1) = \gamma - \gamma^2 k_y^2/k_x^2. \quad (67)$$

The solution is regular for any positive integer m , which we may regard as the (discrete) vertical wavenumber. For given k_x , k_y , and m , equations (65)–(67) implicitly provide a dispersion relation for λ . In general, perturbations with large wavenumbers k_x, k_y are purely oscillatory, and perturbations with sufficiently small k_x, k_y are either stable or unstable. The fastest growing unstable mode has $m = \beta = k_y = 0$, and is of the form

$$\delta\mathbf{B} \propto \begin{pmatrix} \mp i\sqrt{2}\tanh(z) \\ 1 \\ 1 \end{pmatrix} \text{sech}^\alpha(z) \exp(\frac{\alpha^2}{2}t \pm i\frac{\alpha}{\sqrt{2}}x), \quad (68)$$

where $\alpha(\alpha + 1) = \gamma$. We note that the fastest growing mode has $k_y = 0$ (i.e., $\mathbf{J} \cdot \mathbf{k} = 0$) and is therefore non-oscillatory, in accordance with Equation (63).

1. *Limiting cases*

With the choice of profiles $B(z) = n(z) = \text{sech}^\gamma(z)$ used above, the electron velocity profile is $v = -B'/n = \gamma \tanh(z)$, and so the electron shear profile, $v' = \gamma \text{sech}^2(z)$, is either wider or narrower than the $B(z)$ and $n(z)$ profiles, depending on whether $\gamma > 2$ or $\gamma < 2$. The fastest-growing eigenmode (68) has a characteristic lengthscale $\simeq 1/\alpha$ in the x and z directions, which is always at least as wide as the $B(z)$ and $n(z)$ profiles, but can be wider or narrower than the $v'(z)$ profile, depending on γ .

By taking the asymptotic limits $\gamma \rightarrow 0$ and $\gamma \rightarrow \infty$ we obtain two interesting limiting cases. More precisely, noting that $\text{sech}^\gamma(z/\gamma) \rightarrow \exp(-|z|)$ as $\gamma \rightarrow 0$, we deduce from Equation (68) that the fastest growing mode for the background with $B = n = \exp(-|z|)$ has $\delta B_z = \exp(-|z|) \exp(\frac{1}{2}t \pm i\frac{1}{\sqrt{2}}x)$. In this limit the electron velocity becomes a Heaviside function of z , but the lengthscale of the unstable mode remains finite. Conversely, using $\text{sech}^\gamma(z/\gamma^{1/2}) \rightarrow \exp(-z^2/2)$ as $\gamma \rightarrow \infty$, we deduce that the fastest growing mode for the background with $B = n = \exp(-z^2/2)$ has $\delta B_z = \exp(\frac{1}{2}t \pm i\frac{1}{\sqrt{2}}x)$. In this case, the unstable mode has finite amplitude throughout the domain, even where $B(z)$ and $n(z)$ are exponentially small. This case is somewhat pathological, however, because the electron velocity, $v = z$, is unbounded, and the unstable modes form a continuous spectrum.

2. The nature of the instability

It is tempting to think of the density-shear instability as a trapped whistler mode that is advected and sheared by the electron flow, as described by Equation (12). Whistler waves are right-hand polarized, so the displacement vector, ξ , rotates in a sense that depends on the direction of the magnetic field. If the electron shear acts in opposition to this rotation, as illustrated in Figure 1, then the perturbation will be locally amplified. Mathematically, this corresponds to the first term on the right-hand side of Equation (62) having negative sign. This is essentially the physical mechanism proposed by RG02 for their Hall drift instability (see also Refs. 8 and 30).

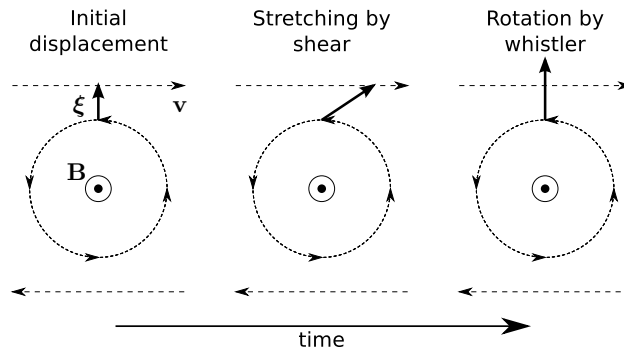


FIG. 1. Cartoon of the density-shear instability. The displacement vector ξ is stretched by the electron velocity \mathbf{v} and rotated about the direction of the magnetic field \mathbf{B} . As a result the initial perturbation is amplified.

However, this physical picture of the instability is incomplete, because it makes no mention of the electron density gradient, which we have previously shown is necessary for instability. (Nor can it explain the resistive tearing instability described in §III B, because it makes no mention of either resistivity or the magnetic null surface.) The role of the density gradient can be better understood by analogy with magneto-buoyancy instability in regular MHD. To that end, suppose we have a layer of magnetic field $\mathbf{B} = B(z) \mathbf{e}_x$ in an MHD fluid with density $n = n(z)$ and a gravitational potential $\phi = \phi(z)$. If we seek marginally stable ($\lambda = 0$) perturbations of the form given by Equation (19), then we eventually arrive at an equation⁴²

$$\frac{b_z''}{b_z} = \frac{B''}{B} + \left(1 + \frac{k_y^2}{k_x^2}\right) \left[k_x^2 - \frac{\phi'}{|\mathbf{v}_A|^2} \left(\frac{n'}{n} + \frac{\phi'}{a^2} \right) \right], \quad (69)$$

where \mathbf{v}_A is the Alfvén velocity, and a is the sound speed. Assuming that the background state is an adiabatic, hydrostatic balance between fluid pressure, magnetic pressure, and gravity, it follows that

$$\frac{n'}{n} + \frac{\phi'}{a^2} = -\frac{|\mathbf{v}_A|^2}{a^2} \frac{B'}{B}, \quad (70)$$

and so Equation (69) can equivalently be written as

$$\frac{b_z''}{b_z} = \frac{B''}{B} + \left(1 + \frac{k_y^2}{k_x^2}\right) \left[k_x^2 - \frac{B'}{B} \left(\frac{n'}{n} + \frac{|\mathbf{v}_A|^2}{a^2} \frac{B'}{B} \right) \right]. \quad (71)$$

The necessary and sufficient condition for magneto-buoyancy instability is that this equation has bounded solutions. Taking the limit $k_y \rightarrow \infty$, bounded solutions can always be obtained if the expression in square brackets is negative for some range of z . There may also be bounded solutions in the opposite limit, $k_y \rightarrow 0$, but the instability criterion is more stringent in that case. In this second limit, Equation (71) becomes almost identical to the density-shear instability equation (62) with $\lambda = k_y = 0$, except that it contains an additional term involving the ratio $|\mathbf{v}_A|/a$, which accounts for the expansion of rising fluid parcels. Such a term does not arise in the density-shear instability because the fluid motions are subject to the constraint given by Equation (46), and so the effective sound speed in EMHD is infinite. If such a constraint were imposed in the magneto-buoyancy problem, then the stability criterion for modes with $k_y = 0$ would exactly match that for density-shear instability. Indeed, in MHD this constraint is called the “anelastic approximation”⁴³, and so the density-shear instability in EMHD is closely analogous to the magneto-buoyancy instability in an anelastic fluid. This connection between EMHD and anelastic MHD is

the natural generalization of the results presented in §III C, as we demonstrate in the next section.

B. Connection with anelastic MHD

The (diffusionless) anelastic equations for a barotropic fluid are⁴³

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{J} \times \mathbf{B}/n - \nabla(P/n + \phi) \quad (72)$$

$$\nabla \cdot (n\mathbf{v}) = 0 \quad (73)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (74)$$

where \mathbf{v} is now the fluid velocity, P is the fluid pressure, ϕ is the gravitational potential, and $n(\mathbf{x})$ is the fluid density, which is a prescribed function of position in the anelastic approximation. We also make Cowling's approximation, in which ϕ is taken to be a fixed function of position. Suppose we have a steady, static equilibrium with $\mathbf{v} = 0$ and

$$\begin{aligned} 0 &= \mathbf{J} \times \mathbf{B}/n - \nabla(P/n + \phi) \\ \Rightarrow 0 &= \nabla \times (\mathbf{J} \times \mathbf{B}/n). \end{aligned} \quad (75)$$

Linear perturbations to this state, when expressed in terms of the Lagrangian fluid perturbation $\boldsymbol{\xi}$, obey the equations

$$n \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathcal{F}(\boldsymbol{\xi}) - n \nabla(\delta P/n) \quad (76)$$

$$\nabla \cdot (n\boldsymbol{\xi}) = 0, \quad (77)$$

where \mathcal{F} is the linear operator defined in Equation (50). As in §III C, it can be shown that \mathcal{F} is a self-adjoint operator over the space of vector fields satisfying Equation (77) with respect to the inner product defined in Equation (51). It follows by an energy argument similar to that of Ref. 40 that the necessary and sufficient condition for instability in this system is that there exists a perturbation $\boldsymbol{\xi}$ that satisfies Equation (77) and has $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle > 0$.

We now return to the EMHD equations (45) and (46). We first note that the anelastic equilibrium condition (75) is also the EMHD equilibrium condition, provided that we now interpret n as the electron density, and ϕ as the electric potential. In this EMHD equilibrium, the Lorentz force is balanced by a combination of the Coulomb force and electron pressure,

rather than by gravity and fluid pressure. It can be shown, in the same manner as in §III C, that $\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle$ is still a conserved quantity in EMHD with non-uniform density, and must therefore vanish for any unstable eigenmode. So the necessary and sufficient condition for instability in anelastic MHD is also (but only) a necessary condition for instability in EMHD. Finally, the generalization of Equation (60) in this case is

$$\langle \boldsymbol{\xi}, \mathcal{F}(\boldsymbol{\xi}) \rangle = - \int dV \left[|\mathbf{B} \cdot \nabla \boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \nabla \ln n) \mathbf{B}|^2 + n \xi_i \frac{\partial^2 \psi}{\partial x_i \partial x_j} \xi_j^* - \frac{1}{2} |\mathbf{B}|^2 n \xi_i \frac{\partial^2 (1/n)}{\partial x_i \partial x_j} \xi_j^* \right], \quad (78)$$

where the function ψ is now defined such that

$$\mathbf{B} \cdot \nabla \mathbf{B} = n \nabla \psi + \frac{1}{2} |\mathbf{B}|^2 \nabla \ln n. \quad (79)$$

This connection between EMHD and anelastic MHD implies that for any instability in the former there must be a corresponding instability in the latter. The connection between the EMHD density-shear instability and the anelastic magneto-buoyancy instability is just one such example.

V. SUMMARY AND DISCUSSION

Most known instabilities in EMHD require either finite conductivity or finite electron inertia. We have demonstrated the existence of a very different instability that, instead, requires electron density gradients and electron velocity shear. This density-shear instability may play an important role in the evolution of magnetic fields in the crusts of neutron stars, as well as in nearly-collisionless plasmas on scales smaller than the ion skin depth. The possibility of such an instability was first recognized in Ref. 3 in the context of laboratory fusion devices, and has received very little attention in other contexts. The instability grows on the Hall timescale, which is generally faster than the growth-rate of any resistive tearing instability, such as the so-called Hall drift instability of RG02. Moreover, the density-shear instability does not require peculiar magnetic field configurations or boundary conditions to operate.

It is highly likely that the instabilities observed in Ref. 30, in a numerical model of a neutron star crust, include both the resistive tearing instability and the density-shear instability. Although they interpreted all of their results in terms of the Hall drift instability, in fact the instabilities they described clearly form two distinct families. One family occurs

close to magnetic null surfaces and has a growth-rate that depends on resistivity, as we would expect for a resistive tearing instability. The other family occurs in the region where density and magnetic pressure gradients are parallel and has a growth-rate that is independent of resistivity, as we would expect for the density-shear instability.

We have demonstrated a connection between the stability properties of EMHD and those of incompressible/anelastic MHD, which shows the futility in seeking EMHD instabilities for field configurations that are already known to be stable in MHD. The density-shear instability is analogous to magneto-buoyancy instability in an anelastic fluid. Under this analogy, the EMHD fluid, which has zero mass and finite charge, becomes an anelastic fluid, which has finite mass and zero charge, and the electric potential becomes the gravitational potential. We have not found any instabilities in ideal, inertia-less EMHD with uniform density; at present, it is not known whether any such instabilities exist.

By analogy with magneto-buoyancy instability in the solar interior⁴⁴ we suggest that the density-shear instability will greatly enhance the transport of magnetic flux from the superconducting core of a neutron star to its surface. This could explain the rapid decrease in the magnetic field strengths observed in young neutron stars^{45,46}. This might also be the explanation for the magnetic spots suggested by Ref. 47.

Of course, the EMHD equilibrium states and magnetic field geometries considered in this paper are rather idealized, and the true situation in neutron star crusts is surely more complex. A full appreciation of the relevance of these results to neutron stars can only come from more realistic, direct numerical simulations, such as those of Refs. 24, 28, and 47.

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